MULTIPLE COMPARISON TESTS FOR CONTRASTS AMONG CORRELATED CORRELATION COEFFICENTS: THE MULTIVARIATE PREDICTOR CASE

Bradley E. Huitema, Western Michigan University

Multiple comparison procedures for testing differences among correlation coefficients have recently been proposed for the independent sample case (Marascuilo, 1966; Huitema, 1974) as well as the correlated sample case (Huitema, 1975). A disadvantage of the procedures proposed for the latter case is that they are all conditional tests. The results of these tests generalize to a population in which the values of the predictor variables are fixed at the values included in the sample. In most correlation research interest lies in the generalization of results to future observed values of the predictor variables. That is, we wish to generalize the results to the whole range of predictor values included in the population from which the sample values were randomly selected rather than to a population including only the predictor values actually included in the sample. Methods of dealing with this problem in the case of two predictor variables have been proposed. The purpose of this paper is to suggest two methods of dealing with the multivariate normal case in which three or more predictors are involved and multiple comparisons among the correlated correlation coefficients are required.

Method I

Method I is proposed for tests on all pairwise comparisons. This method involves two stages:

Stage A: The overall hypothesis

$$H_0: \rho_{yx_1} = \rho_{yx_2} = \cdots = \rho_{yx_m}$$

(where ρ is the population correlation coefficient, y is the dependent variable and x_1 through x_m are the predictor variables) is tested using a generalization of Hotelling's well known test of the equality of two correlated correlation coefficients (Hotelling, 1940). If this test is significant at the specified level of alpha, proceed to stage B.

Stage B: Compute Williams' modification of Hotelling's two predictor test for each pairwise comparison using the same alpha employed during Stage A.

Computation Procedure

The computation procedure for Stages A and B are described in this section.

Stage A

Step 1

Compute $\underline{R}_x$, the interrcorrelation matrix of all predictor variables.

Step 2

Compute $\underline{R}^{-1}_x$, the inverse of $\underline{R}_x$.

Step 3

Compute $\underline{r}_{yx}$, the column vector of correlations between the dependent variable and each predictor variable, i.e.,

$$\underline{r}_{yx} = \begin{bmatrix} r_{yx_1} \\ r_{yx_2} \\ \cdot \\ \cdot \\ \cdot \\ r_{yx_m} \end{bmatrix}$$

Step 4

Compute the coefficient of multiple determination $R^2_{yx_1x_2\cdots x_m}$.

Step 5

Obtain $\sum\sum c_{ij}$, the sum of the elements of $\underline{R}^{-1}_x$.

Step 6

Obtain $c_1, c_2, \ldots c_m$, the sums of rows 1 through m of $\underline{R}^{-1}_x$.

Step 7

Compute the weights $w_1, w_2, \ldots w_m$ where the ith weight is

$$w_i = \frac{c_i}{\sum\sum c_{ij}}.$$

The sum of the $w_i = 1$. The column vector of weights is denoted $\underline{w}$.

Step 8

Compute $\underline{w}' \, \underline{r}_{yx} = h$.

Step 9

Compute the product $h^2(\sum\sum c_{ij})$.

Step 10

Compute the F statistic using

$$F = \frac{\left[R^2_{yx_1x_2 \ldots x_m} - h^2(\sum\sum c_{ij})\right] / m-1}{(1 - R^2_{yx_1x_2\ldots x_m})/N-m-1}.$$

The obtained F is evaluated with $F_{\alpha, m-1, N-m-1}$ where N is the total number of subjects and m is the number of predictor variables.

Stage B

Williams' modification of Hotelling's two predictor test is distributed approximately as F with 1 and N-3 degrees of freedom; it can be written as follows:

$$\frac{(r_{yx_i} - r_{yx_j})^2}{2\left[(1-R^2_{yx_ix_j})/N-3\right](1-r_{x_ix_j}) + \frac{(r_{yx_i} + r_{yx_j})^2 (1 - r_{x_ix_j})^3}{4\,(N-1)\,(1 + r_{x_ix_j})}}$$

where r_{yx_i} and r_{yx_j} are the sample correlations between the dependent variable and the ith and jth predictor variables respectively.

N is the total number of subjects, $R^2_{yx_ix_j}$ is the sample coefficient of multiple determination between the dependent variable and the ith and jth predictor variables and $r_{x_ix_j}$ is the sample correlation between the ith and jth predictor variables.

Example

Several members of the faculty of the W.M.U. department of psychology were interested in the relationship between achievement in the required statistics course and performance on each of three predictor variables (G.R.E. - Verbal, G.R.E. - Quantitative and M.A.T.) sometimes used in the selection of graduate students. Measures on y (incorrect response on seven exams) and the three predictors x_1, x_2 and x_3 were obtained from each subject in a sample of 51 graduate students. The three predictor - criterion correlations were -.22, -.71, and -.01. Tests among these correlations follow:

Stage A

Step 1

$$\underline{R}_x = \begin{bmatrix} 1 & .311746 & .695068 \\ .311746 & 1 & .275878 \\ .695068 & .275878 & 1 \end{bmatrix}$$

Step 2

$$\underline{R}_x^{-1} \begin{bmatrix} 1.994826 & -.259082 & -1.315064 \\ -2.59082 & 1.116027 & -.127808 \\ -1.315064 & -.127808 & 1.949318 \end{bmatrix}$$

Step 3

$$\underline{r}_{yx} \begin{bmatrix} -.2247784 \\ -.7079091 \\ -.0113289 \end{bmatrix}$$

Step 4

$$R^2_{yx_1x_2x_3} = .56912087$$

Step 5 $\sum\sum c_{ij} = 1.656263$

Step 6

$c_1 = .42068$

$c_2 = .729137$

$c_3 = .506446$

Step 7

$$w_1 = \frac{c_1}{\sum\sum c_{ij}} = .25399935$$

$$w_2 = \frac{c_2}{\sum\sum c_{ij}} = .44023021$$

$$w_3 = \frac{c_3}{\sum\sum c_{ij}} = .30577632$$

Step 8

$$h = \begin{bmatrix} .25399935 & .44023021 & .30577632 \end{bmatrix} \begin{bmatrix} -.2247784 \\ -.7079091 \\ -.0113289 \end{bmatrix}$$

$$= .37220065$$

Step 9

$$h^2\sum\sum c_{ij} = .13853332\ (1.656263) = .229448$$

Step 10

$$F = \frac{(.3396733)/2}{(.4308792)/47} = 18.53$$

Since the obtained value of F exceeds the critical value of F (which is 5.11 for alpha = .01 using 2 and 47 degrees of freedom) we reject the overall hypothesis

$$\rho_{yx_1} = \rho_{yx_2} = \rho_{yx_3}$$

and proceed to Stage B.

Stage B

Pairwise comparisons among the three predictor-criterion correlations are computed as follows:

$$\boxed{H_0: \rho_{yx_1} = \rho_{yx_2}}$$

$$F = \frac{\left[(-.2247784)-(-.7079091)\right]^2}{2\left[(1-.50115)/48\right](1-.3117462) + \left[\frac{\left[(-.2247784)+(-.7079091)\right]^2 (1-.3117462)^3}{4\,(51-1)(1.3117462)}\right]}$$

$$= \frac{.2334152}{.0143056}$$

$$= 15.17$$

$$\boxed{H_0: \rho_{yx_1} = \rho_{yx_3}}$$

$$F = \frac{\left[(-.2247784)-(-.0113289)\right]^2}{2\left[(1-.09115)/48\right](1-.6950676) + \left[\frac{\left[(-.2247784)+(-.0113289)\right]^2 (1-.6940676)^3}{4(51-1)(1+ .6950676)}\right]}$$

$$= \frac{.0455606}{.0115519}$$

$$= 3.94$$

$$\boxed{H_0: \rho_{yx_2} = \rho_{yx_3}}$$

$$F = \frac{\left[(-.7079091)-(-.0113289)\right]^2}{2\left[(1-.53777)/48\right](1-.4758782) + \left[\frac{\left[(-.7079091)+(-.0113289)\right]^2 (1-.2758782)^3}{4\,(51-1)\,(1.2758782)}\right]}$$

$$= \frac{.4852239}{.0147158}$$

$$= 32.97$$

The three pairwise F tests yield values of 15.17, 3.94 and 32.97. These obtained values are compared with the critical value of F based on alpha = .01 with one and 48 degrees of freedom. Since the critical value is 7.25 we reject H_0: $\rho_{yx_1} = \rho_{yx_2}$ and H_0: $\rho_{yx_2} = \rho_{yx_3}$ and retain H_0: $\rho_{yx_1} = \rho_{yx_3}$.

Method II

The second method is proposed for the situation in which a relatively large number of predictor-criterion correlations is involved but the researcher has interest in making only a few planned comparisons. Unlike Method I, no preliminary overall test is run. Instead, each planned comparison is tested using Williams' modification of Hotelling's two predictor test (i.e., Stage B of Method I). The obtained F values are not, however, compared with the conventional points of the F distribution. Rather, the obtained F values are compared with the critical value of the Bonferroni F statistic associated with C (the number of planned comparisons) and one and N-3 degrees of freedom.

Example

If Method II is applied to the three correlations employed in the example previously described, we simply compare the obtained F values of 15.17, 3.94, and 32.97 with the tabled value of the Bonferroni F statistic (available in Huitema, In Press).

If all three pairwise contrasts are planned, we find the critical value is 9.59 for alpha (family) = .01. (If only two of the three possible pairwise contrasts had been planned we find the critical value is 8.69). Note that the decisions concerning the comparisons are the same (for these data) with Methods I and II.

Discussion

An important consideration in the application of Methods I and II is the experimentwise error rate. The probability of one or more false rejections in a study with m contrasts can be expected to be equal to or less than the nominal alpha with Method I if the Stage A test maintains the experimentwise error rate at the nominal alpha. Since the characteristics of the m predictor generalization of Hotelling's test (i.e., Stage A) have not been investigated, a clear statement of the error rate associated with this method is not currently possible. However, studies of the per comparison error rate associated with Hotelling's two predictor test suggest that the error rate is greater than alpha with certain correlation matrices. Hendrickson, Stanley and Hills (1970) and Hendrickson and Collins (1970) compared the empirical results of Hotelling's test with the results obtained using Williams' modification of Hotelling's test and Olkin's test. Both Williams' and Olkin's procedures were designed for the trivariate normal case rather than the fixed predictor case. The three procedures yielded practically the same results in twelve similar empirical studies. On the other hand, a very extensive Monte Carlo study carried out by Neill and Dunn (1975) led them to conclude that Hotelling's test is completely unsatisfactory because it fails to control Type I error near the nominal value with certain correlation matrices. The empirical Type I error rate was found to be as high as .92 when a nominal alpha value of .05 was employed. Williams' test maintained the empirical error rate very close to the nominal value regardless of the correlation matrix employed. Unfortunately, Neill and Dunn did not include Olkin's test in their study.

Since the m predictor generalization of Hotelling's test may also yield higher than nominal Type I errors with certain correlation matrices, the first stage of Method I may allow too many contrasts to reach Stage B. This will then lead to too many errors at Stage B to maintain the experimentwise error rate at the nominal value.

The experimentwise error rate associated with Method II is less than alpha because (a) Williams' test maintains the per comparison error rate at alpha and (b) the Bonferroni inequality makes it clear that the alpha associated with the whole collection of contrasts can not be greater than the sum of the individual alphas, i.e.,

$$\alpha_{\text{experimentwise}} \leq \sum_{i=1}^{m} \alpha_{\text{individual}} .$$

Since it is not yet clear whether or not Stage A of Method I sufficiently controls experimentwise error, the conservative approach is to employ Method II. Monte Carlo studies of the power and error rate associated with a large variety of correlation matrices are needed for both methods.

References

Hendrickson, G. F., and Collins, J. R. Note correcting the results in "Olkin's New Formula for the significance of r13 vs. r23 compared with Hotelling's method." American Educational Research Journal, 1970, 7, 639-641.

Hendrickson, G. F., Stanley, J. C., and Hills, J. R. Olkin's new formula for significance of r13 vs. r23 compared with Hotelling's method. American Educational Research Journal, 1970, 7, 189-195.

Hotelling, H. The selection of variates for use in prediction with some comments on the general problem of nuisance parameters. Annals of Mathematical Statistics, 1940, 11, 271-283.

Huitema, B. E. Three multiple comparison procedures for contrasts among correlation coefficients. Paper presented at the annual meeting of the American Statistical Association, St. Louis, 1974.

Huitema, B. E. Multiple comparison tests for contrasts among correlated correlation coefficients. Paper presented at the annual meeting of the American Statistical Association, Atlanta, 1975.

Huitema, B. E. The Analysis of Covariance and Alternatives. New York: Wiley, In press.

Marascuilo, L. A. Large-sample multiple comparisons. Psychological Bulletin, 1966, 65, 280-290.

Neill, J. J., and Dunn, D. J. Equality of dependent correlation coefficients. Biometrics, 1975, 31, 531-543.

Williams, E. J. The comparison of regression variables. Journal of the Royal Statistical Society - Series B, 1959, 21, 396-399.